Q.4 Show that $\vec{\nabla} r^n = n r^{n-2} \vec{r}$; where $\vec{r}$ is a position vector.

A: $\vec{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}| = (x^2+y^2+z^2)^{1/2}$

Now $\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}$

$= \hat{i}\,\frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot \cancel{2}x$

$= \hat{i}\, nx(x^2+y^2+z^2)^{\frac{n}{2}-1}$

Similarly; $\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} = \hat{j}\, ny(x^2+y^2+z^2)^{\frac{n}{2}-1}$

$\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} = \hat{k}\, nz(x^2+y^2+z^2)^{\frac{n}{2}-1}$

$\therefore \vec{\nabla} r^n = n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i}+y\hat{j}+z\hat{k})$

$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\,\vec{r}$

$= \cancel{n(\sqrt{x}} = n r^{n-2}\,\vec{r}$ Proved

Q. Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

A: $\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4xz^3 - 3x^2y^2z)$

$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$.

At the point $(2,-1,2)$;

$\vec{\nabla}\varphi = (32-24)\hat{i} + 48\hat{j} + (96-48)\hat{k}$

$= \cancel{8\hat{i} + 48\hat{j} + 48\hat{k}}\; 8\hat{i} + 48\hat{j} + 84\hat{k}$.

Now unit vector in the direction of $2\hat{i} - 3\hat{j} + 6\hat{k}$ is

$$\hat{a} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{1}{7}(2\hat{i} - 3\hat{j} + 6\hat{k})$$

Then the required directional derivative is

$$\vec{\nabla}\phi \cdot \hat{a} = (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{1}{7}(2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$= \frac{1}{7}(16 - 144 + 504) = \frac{376}{7} \text{ Ans:}$$

Q.5. Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$, where c is a constant.

A:- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a position vector to any point $P(x,y,z)$ on the surface.

$$\phi(x,y,z) = c$$
$$d\phi = 0.$$

$$\boxed{\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ d\vec{r} &= dx\hat{i} + dy\hat{j} + dz\hat{k}\end{aligned}}$$

But $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$$

$$\text{or}\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0$$

$$\text{or, } \vec{\nabla}\phi \cdot \vec{dr} = 0$$

That is $\vec{\nabla}\phi$ is perpendicular to $\vec{dr}$, and therefore $\vec{\nabla}\phi$ is perpendicular to the surface.

Q.70 / P-79. If $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$; $\phi = 3x^2 - yz$, find (i) $\vec{A}\cdot\vec{\nabla}\phi$ (ii) $\vec{\nabla}\cdot(\vec{\nabla}\phi)$ at the point $(1,-1,1)$.

(i) $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2 - yz)$

$= 6x\hat{i} - z\hat{j} - y\hat{k}$.

$\therefore \vec{A}\cdot\vec{\nabla}\phi = (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k})\cdot(6x\hat{i} - z\hat{j} - y\hat{k})$

$= 18x^2yz^2 - 2xy^3z + x^2y^2z$.

At the point $(1,-1,1)$; $\vec{A}\cdot\vec{\nabla}\phi = -18 + 2 + 1 = -15$ Ans:

(ii) $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(6x\hat{i} - z\hat{j} - y\hat{k})$

$= \frac{\partial}{\partial x}(6x) - \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)$

$= 6 - 0 - 0 = 6$ Ans:

curl :

Q-89 / P-80. If $\vec{A} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ & $\phi = x^2yz$

find (i) $\vec{\nabla}\times\vec{A}$ (ii) ~~or~~ $\vec{\nabla}\times(\phi\vec{A})$ at the point $(1,1,1)$.

Ans: $\vec{A} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$$\therefore \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(3xz^3) + \frac{\partial}{\partial z}(yz)\right\}\hat{i} - \left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\}\hat{j}$$
$$+ \left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}\hat{k}$$

$$\vec{\nabla}\times\vec{A} = (0+y)\hat{i} - (3z^3 - 4xz)\hat{j} + (-0-0)\hat{k}$$
$$= y\hat{i} - (3z^3 - 4xz)\hat{j}$$

At the point $(1,1,1)$; $\vec{\nabla}\times\vec{A} = \hat{i} + \hat{j}$ Ans:

(ii) $\varphi = x^2yz$; $\therefore \varphi\vec{A} = 2x^3yz^3\hat{i} - x^2y^2z^2\hat{j} + 3x^3yz^4\hat{k}$.

$$\therefore \vec{\nabla}\times(\varphi\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^3yz^3 & -x^2y^2z^2 & 3x^3yz^4 \end{vmatrix}$$

After evaluation;
$$\vec{\nabla}\times(\varphi\vec{A}) = (3x^3z^4 + 2x^2y^2z)\hat{i} - (9x^2yz^4 - 6x^3yz^2)\hat{j} + (-2xy^2z^2 - 2x^3z^3)\hat{k}$$

Now $\vec{\nabla}\cdot\vec{\nabla}\times(\varphi\vec{A})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(3x^3z^4 + 2x^2y^2z)\hat{i} - (9x^2yz^4 - 6x^3yz^2)\hat{j} - (2xy^2z^2 + 2x^3z^3)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(3x^3z^4 + 2x^2y^2z) - \frac{\partial}{\partial y}(9x^2yz^4 - 6x^3yz^2) - \frac{\partial}{\partial z}(2xy^2z^2 + 2x^3z^3)$$

$$= \cancel{9x^2z^4} + \cancel{4xy^2z} - \cancel{9x^2z^4} + \cancel{6x^3z^2} - \cancel{4xy^2z} - \cancel{6x^3z^2}$$

$$= 0$$

[Note: Div (curl of any vector) $=0$]

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Q: Prove that $\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Ans :- $\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}\right)$

$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$

$= 0 + 0 - 0 = 0$

$\therefore \vec{A}$ is solenoidal.

Q: Show that $\vec{F} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ is irrotational.

Ans: $\vec{F} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$

Now
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right\}\hat{i} - \left\{\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial z}(4xy - z^3)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right\}\hat{k}$$

$= (-0 - 0)\hat{i} - (-3z^2 + 3z^2)\hat{j} + (4x - 4x)\hat{k}$

$= 0 - 0 + 0 = 0$

$\therefore \vec{F}$ is Irrotational.

Note: If $\vec{\nabla}\times\vec{F} = 0$; $\vec{F}$ is irrotational. In this case $\vec{F}$ is also conservative force ~~field~~.

<u>conservative force</u>: A conservative force is that work done by it is independent of the path and depends only the initial & final position. In nature; gravitational force, magnetic force, electrostatic force etc.